

The Alperin-McKay Conjecture for metacyclic, minimal non-abelian defect groups

Benjamin Sambale

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Abstract

We prove the Alperin-McKay Conjecture for all p -blocks of finite groups with metacyclic, minimal non-abelian defect groups. These are precisely the metacyclic groups whose derived subgroup have order p . In the special case $p = 3$, we also verify Alperin's Weight Conjecture for these defect groups. Moreover, in case $p = 5$ we do the same for the non-abelian defect groups $C_{25} \rtimes C_{5^n}$. The proofs do *not* rely on the classification of the finite simple groups.

Keywords: Alperin-McKay Conjecture, metacyclic defect groups

AMS classification: 20C15, 20C20

1 Introduction

Let B be a p -block of a finite group G with respect to an algebraically closed field of characteristic p . Suppose that B has a metacyclic defect group D . We are interested in the number $k(B)$ (respectively $k_i(B)$) of irreducible characters of B (of height $i \geq 0$), and the number $l(B)$ of irreducible Brauer characters of B . If $p = 2$, these invariants are well understood and the major conjectures are known to be true by work of several authors (see [4, 31, 35, 37, 11, 9]). Thus we will focus on the case $p > 2$ in the present work. Here at least Brauer's $k(B)$ -Conjecture, Olsson's Conjecture and Brauer's Height Zero Conjecture are satisfied for B (see [14, 43, 38]). By a result of Stancu [40], B is a controlled block. Moreover, if D is a non-split extension of two cyclic groups, it is known that B is nilpotent (see [7]). Then a result by Puig [33] describes the source algebra of B in full detail. Thus we may assume in the following that D is a split extension of two cyclic groups. A famous theorem by Dade [6] handles the case where D itself is cyclic by making use of Brauer trees. The general situation is much harder – even the case $D \cong C_3 \times C_3$ is still open (see [24, 42, 26, 25]). Now consider the subcase where D is non-abelian. Then a work by An [1] shows that G is not a quasisimple group. On the other hand, the algebra structure of B in the p -solvable case can be obtained from Külshammer [27]. If B has maximal defect (i. e. $D \in \text{Syl}_p(G)$), the block invariants of B were determined in [15]. If B is the principal block, Horimoto and Watanabe [20] constructed a perfect isometry between B and its Brauer correspondent in $N_G(D)$.

Let us suppose further that D is a split extension of a cyclic group and a group of order p (i. e. D is the unique non-abelian group with a cyclic subgroup of index p). Here the difference $k(B) - l(B)$ is known from [16]. Moreover, under additional assumptions on G , Holloway, Koshitani and Kunugi [19] obtained the block invariants precisely. In the special case where D has order p^3 , incomplete information are given by Hendren [17]. Finally, one has full information in case $|D| = 27$ by work of the present author [38].

In the present work we consider the following class of non-abelian split metacyclic groups

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, yxy^{-1} = x^{1+p^{m-1}} \rangle \cong C_{p^m} \rtimes C_{p^n} \quad (1.1)$$

where $m \geq 2$ and $n \geq 1$. By a result of Rédei (see [21, Aufgabe III.7.22]) these are precisely the metacyclic, minimal non-abelian groups. A result by Knoche (see [21, Aufgabe III.7.24]) implies further that these are exactly the metacyclic groups with derived subgroup of order p . In particular the family includes the non-abelian group with a cyclic subgroup of index p mentioned above. The main theorem of the present paper states

that $k_0(B)$ is locally determined. In particular the Alperin-McKay Conjecture holds for B . This improves some of the results mentioned above. We also prove that every irreducible character of B has height 0 or 1. This is in accordance with the situation in $\text{Irr}(D)$. In the second part of the paper we investigate the special case $p = 3$. Here we are able to determine $k(B)$, $k_i(B)$ and $l(B)$. This gives an example of Alperin's Weight Conjecture and the Ordinary Weight Conjecture. Finally, we determine the block invariants for $p = 5$ and $D \cong C_{25} \rtimes C_{5^n}$ where $n \geq 1$.

As a new ingredient (compared to [38]) we make use of the focal subgroup of B .

2 The Alperin-McKay Conjecture

Let p be an odd prime, and let B be a p -block with split metacyclic, non-abelian defect group D . Then D has a presentation of the form

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, yxy^{-1} = x^{1+p^l} \rangle$$

where $0 < l < m$ and $m - l \leq n$. Elementary properties of D are stated in the following lemma.

Lemma 2.1.

- (i) $D' = \langle x^{p^l} \rangle \cong C_{p^{m-l}}$.
- (ii) $Z(D) = \langle x^{p^{m-l}} \rangle \times \langle y^{p^{m-l}} \rangle \cong C_{p^l} \times C_{p^{n-m+l}}$.

Proof. Omitted. □

We fix a Sylow subpair (D, b_D) of B . Then the conjugation of subpairs $(Q, b_Q) \leq (D, b_D)$ forms a saturated fusion system \mathcal{F} on D (see [2]). Here $Q \leq D$ and b_Q is a uniquely determined block of $C_G(Q)$. We also have subsections (u, b_u) where $u \in D$ and $b_u := b_{\langle u \rangle}$. By [40], \mathcal{F} is controlled. Moreover by Theorem 2.5 in [14] we may assume that the inertial group of B has the form $N_G(D, b_D)/C_G(D) = \text{Aut}_{\mathcal{F}}(D) = \langle \text{Inn}(D), \alpha \rangle$ where $\alpha \in \text{Aut}(D)$ such that $\alpha(x) \in \langle x \rangle$ and $\alpha(y) = y$. By a slight abuse of notation we often write $\text{Out}_{\mathcal{F}}(D) = \langle \alpha \rangle$. In particular the inertial index $e(B) := |\text{Out}_{\mathcal{F}}(D)|$ is a divisor of $p - 1$. Let

$$\text{foc}(B) := \langle f(a)a^{-1} : a \in Q \leq D, f \in \text{Aut}_{\mathcal{F}}(Q) \rangle$$

be the *focal subgroup* of B (or of \mathcal{F}). Then it is easy to see that $\text{foc}(B) \subseteq \langle x \rangle$. In case $e(B) = 1$, B is nilpotent and $\text{foc}(B) = D'$. Otherwise $\text{foc}(B) = \langle x \rangle$.

For the convenience of the reader we collect some estimates on the block invariants of B .

Proposition 2.2. *Let B be as above. Then*

$$\begin{aligned} \left(\frac{p^l + p^{l-1} - p^{2l-m-1} - 1}{e(B)} + e(B) \right) p^n &\leq k(B) \leq \left(\frac{p^l - 1}{e(B)} + e(B) \right) (p^{n+m-l-2} + p^n - p^{n-2}), \\ 2p^n &\leq k_0(B) \leq \left(\frac{p^l - 1}{e(B)} + e(B) \right) p^n, \\ \sum_{i=0}^{\infty} p^{2i} k_i(B) &\leq \left(\frac{p^l - 1}{e(B)} + e(B) \right) p^{n+m-l}, \\ l(B) &\geq e(B) \mid p - 1, \\ p^n \mid k_0(B), \quad p^{n-m+l} \mid k_i(B) &\text{ for } i \geq 1, \\ k_i(B) &= 0 \quad \text{for } i > 2(m-l). \end{aligned}$$

Proof. Most of the inequalities are contained in Proposition 2.1 to Corollary 2.5 in [38]. By Theorem 1 in [36] we have $p^n \mid |D : \text{foc}(B)| \mid k_0(B)$. In particular $p^n \leq k_0(B)$. In case $k_0(B) = p^n$ it follows from [23] that B is nilpotent. However then we would have $k_0(B) = |D : D'| = p^{n+l} > p^n$. Therefore $2p^n \leq k_0(B)$. Theorem 2 in [36] implies $p^{n-m+l} \mid |Z(D) : Z(D) \cap \text{foc}(B)| \mid k_i(B)$ for $i \geq 1$. □

Now we consider the special case where $m = l + 1$. As mentioned in the introduction these are precisely the metacyclic, minimal non-abelian groups. We prove the main theorem of this section.

Theorem 2.3. *Let B be a p -block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime p . Then*

$$k_0(B) = \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n$$

with the notation from (1.1). In particular the Alperin-McKay Conjecture holds for B .

Proof. By Proposition 2.2 we have

$$p^n \mid k_0(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n.$$

Thus, by way of contradiction we may assume that

$$k_0(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) - 1 \right) p^n.$$

We also have

$$k(B) \geq \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B) \right) p^n$$

from Proposition 2.2. Hence the sum $\sum_{i=0}^{\infty} p^{2i} k_i(B)$ will be small if $k_0(B)$ is large and $k_1(B) = k(B) - k_0(B)$. This implies the following contradiction

$$\begin{aligned} \left(\frac{p^m - 1}{e(B)} + p^2 + e(B) - 1 \right) p^n &= \left(\frac{p^{m-1} - 1}{e(B)} + e(B) - 1 \right) p^n + \left(\frac{p^{m-2} - p^{m-3}}{e(B)} + 1 \right) p^{n+2} \\ &\leq \sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \left(\frac{p^m - p}{e(B)} + p e(B) \right) p^n < \left(\frac{p^m - 1}{e(B)} + p^2 \right) p^n. \end{aligned}$$

Since the Brauer correspondent of B in $N_G(D)$ has the same fusion system, the Alperin-McKay Conjecture follows. \square

Isaacs and Navarro [22, Conjecture D] proposed a refinement of the Alperin-McKay Conjecture by invoking Galois automorphisms. We show (as a improvement of Theorem 4.3 in [38]) that this conjecture holds in the special case $|D| = p^3$ of Theorem 2.3. We will denote the subset of $\text{Irr}(B)$ of characters of height 0 by $\text{Irr}_0(B)$.

Corollary 2.4. *Let B be a p -block of a finite group G with non-abelian, metacyclic defect group of order p^3 . Then Conjecture D in [22] holds for B .*

Proof. Let D be a defect group of B . For $k \in \mathbb{N}$, let \mathbb{Q}_k be the cyclotomic field of degree k . Let $|G|_{p'}$ be the p' -part of the order of G . It is well-known that the Galois group $\mathcal{G} := \text{Gal}(\mathbb{Q}_{|G|} | \mathbb{Q}_{|G|_{p'}})$ acts canonically on $\text{Irr}(B)$. Let $\gamma \in \mathcal{G}$ be a p -element. Then it suffices to show that γ acts trivially on $\text{Irr}_0(B)$. By Lemma IV.6.10 in [12] it is enough to prove that γ acts trivially on the \mathcal{F} -conjugacy classes of subsections of B via $\gamma(u, b_u) := (u^{\overline{\gamma}}, b_u)$ where $u \in D$ and $\overline{\gamma} \in \mathbb{Z}$. Since γ is a p -element, this action is certainly trivial unless $|\langle u \rangle| = p^2$. Here however, the action of γ on $\langle u \rangle$ is just the D -conjugation. The result follows. \square

In the situation of Corollary 2.4 one can say a bit more: By Proposition 3.3 in [38], $\text{Irr}(B)$ splits into the following families of p -conjugate characters:

- $(p - 1)/e(B) + e(B)$ orbits of length $p - 1$,
- two orbits of length $(p - 1)/e(B)$,
- at least $e(B)$ p -rational characters.

Without loss of generality, let $e(B) > 1$. By Theorem 4.1 in [38] we have $k_1(B) \leq (p-1)/e(B) + e(B) - 1$. Moreover, Proposition 4.1 of the same paper implies $k_1(B) < p-1$. In particular, all orbits of length $p-1$ of p -conjugate characters must lie in $\text{Irr}_0(B)$. In case $e(B) = p-1$ the remaining $(p-1)/e(B) + e(B)$ characters in $\text{Irr}_0(B)$ must be p -rational. Now let $e(B) < \sqrt{p-1}$. Then it is easy to see that $\text{Irr}_0(B)$ contains just one orbit of length $(p-1)/e(B)$ of p -conjugate characters. Unfortunately, it is not clear if this also holds for $e(B) \geq \sqrt{p-1}$.

Next we improve the bound coming from Proposition 2.2 on the heights of characters.

Proposition 2.5. *Let B be a p -block of a finite group with metacyclic, minimal non-abelian defect groups. Then $k_1(B) = k(B) - k_0(B)$. In particular, B satisfies the following conjectures:*

- *Eaton's Conjecture [8]*
- *Eaton-Moretó Conjecture [10]*
- *Robinson's Conjecture [28, Conjecture 4.14.7]*
- *Malle-Navarro Conjecture [29]*

Proof. By [37] we may assume $p > 2$ as before. By way of contradiction suppose that $k_i(B) > 0$ for some $i \geq 2$. Since

$$k(B) \geq \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B) \right) p^n,$$

we have $k(B) - k_0(B) \geq (p^{m-1} - p^{m-2})p^{n-1}/e(B)$ by Theorem 2.3. By Proposition 2.2, $k_1(B)$ and $k_i(B)$ are divisible by p^{n-1} . This shows

$$\left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n + \left(\frac{p^{m-1} - p^{m-2}}{e(B)} - 1 \right) p^{n+1} + p^{n+3} \leq \sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^{n+1}.$$

Hence, we derive the following contradiction

$$p^{n+3} - p^{n+1} \leq \left(\frac{1-p}{e(B)} + e(B)(p-1) \right) p^n \leq p^{n+2}.$$

This shows $k_1(B) = k(B) - k_0(B)$. Now Eaton's Conjecture is equivalent to Brauer's $k(B)$ -Conjecture and Olsson's Conjecture. Both are known to hold for all metacyclic defect groups. Also the Eaton-Moretó Conjecture and Robinson's Conjecture is trivially satisfied for B . The Malle-Navarro Conjecture asserts that $k(B)/k_0(B) \leq k(D') = p$ and $k(B)/l(B) \leq k(D)$. By Theorem 2.3 and Proposition 2.2, the first inequality reduces to $p^{n-1} + p^n - p^{n-2} \leq p^{n+1}$ which is true. For the second inequality we observe that every conjugacy class of D has at most p elements, since $|D : Z(D)| = p^2$. Hence, $k(D) = |Z(D)| + \frac{|D| - |Z(D)|}{p} = p^{n+m-1} + p^{n+m-2} - p^{n+m-3}$. Now Proposition 2.2 gives

$$\frac{k(B)}{l(B)} \leq k(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) (p^{n-1} + p^n - p^{n-2}) \leq p^{n+m-1} + p^{n+m-2} - p^{n+m-3} = k(D). \quad \square$$

We use the opportunity to present a result for $p = 3$ and a different class of metacyclic defect groups (where $l = 1$ with the notation above).

Theorem 2.6. *Let B be a 3-block of a finite group G with defect group*

$$D = \langle x, y \mid x^{3^m} = y^{3^n} = 1, \, yxy^{-1} = x^4 \rangle$$

where $2 \leq m \leq n+1$. Then $k_0(B) = 3^{n+1}$. In particular, the Alperin-McKay Conjecture holds for B .

Proof. We may assume that B is non-nilpotent. By Proposition 2.2 we have $k_0(B) \in \{2 \cdot 3^n, 3^{n+1}\}$. By way of contradiction, suppose that $k_0(B) = 2 \cdot 3^n$. Let $P \in \text{Syl}_p(G)$. Since $D/\text{foc}(B)$ acts freely on $\text{Irr}_0(B)$, there are 3^n characters of degree $a|P : D|$, and 3^n characters of degree $b|P : D|$ in B for some $a, b \geq 1$ such that $3 \nmid a, b$. Hence,

$$\left| \sum_{\chi \in \text{Irr}_0(B)} \chi(1)^2 \right|_3 = 3^n |P : D|^2 (a^2 + b^2)_3 = |P : D|^2 |D : \text{foc}(B)|.$$

Now Theorem 1.1 in [23] gives a contradiction. \square

A generalization of the argument in the proof shows that in the situation of Proposition 2.2, $k_0(B) = 2p^n$ can only occur if $p \equiv 1 \pmod{4}$.

3 Lower defect groups

In the following we use the theory of lower defect groups in order to estimate $l(B)$. We cite a few results from the literature. Let B be a p -block of a finite group G with defect group D and Cartan matrix C . We denote the multiplicity of an integer a as elementary divisor of C by $m(a)$. Then $m(a) = 0$ unless a is a p -power. It is well-known that $m(|D|) = 1$. Brauer [3] expressed $m(p^n)$ ($n \geq 0$) in terms of 1-multiplicities of lower defect groups (see also Corollary V.10.12 in [12]):

$$m(p^n) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R) \quad (3.1)$$

where \mathcal{R} is a set of representatives for the G -conjugacy classes of subgroups $R \leq D$ of order p^n . Later (3.1) was refined by Broué and Olsson by invoking the fusion system \mathcal{F} of B .

Proposition 3.1 (Broué-Olsson [5]). *For $n \geq 0$ we have*

$$m(p^n) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R, b_R)$$

where \mathcal{R} is a set of representatives for the \mathcal{F} -conjugacy classes of subgroups $R \leq D$ of order p^n .

Proof. This is (2S) of [5]. □

In the present paper we do not need the precise (and complicated) definition of the non-negative numbers $m_B^{(1)}(R)$ and $m_B^{(1)}(R, b_R)$. We say that R is a *lower defect group* for B if $m_B^{(1)}(R, b_R) > 0$. In particular, $m_B^{(1)}(D, b_D) = m_B^{(1)}(D) = m(|D|) = 1$. A crucial property of lower defect groups is that their multiplicities can usually be determined locally. In the next lemma, $b_R^{\text{N}_G(R, b_R)}$ denotes the (unique) Brauer correspondent of b_R in $\text{N}_G(R, b_R)$.

Lemma 3.2. *For $R \leq D$ and $B_R := b_R^{\text{N}_G(R, b_R)}$ we have $m_B^{(1)}(R, b_R) = m_{B_R}^{(1)}(R)$. If R is fully \mathcal{F} -normalized, then B_R has defect group $\text{N}_D(R)$ and fusion system $\text{N}_{\mathcal{F}}(R)$.*

Proof. The first claim follows from (2Q) in [5]. For the second claim we refer to Theorem IV.3.19 in [2]. □

Another important reduction is given by the following lemma.

Lemma 3.3. *For $R \leq D$ we have $\sum_{Q \in \mathcal{R}} m_{B_R}^{(1)}(Q) \leq l(b_R)$ where \mathcal{R} is a set of representatives for the $\text{N}_G(R, b_R)$ -conjugacy classes of subgroups Q such that $R \leq Q \leq \text{N}_D(R)$.*

Proof. This is implied by Theorem 5.11 in [32] and the remark following it. Notice that in Theorem 5.11 it should read $B \in \text{Bl}(G)$ instead of $B \in \text{Bl}(Q)$. □

In the local situation for B_R also the next lemma is useful.

Lemma 3.4. *If $\text{O}_p(Z(G)) \not\leq R$, then $m_B^{(1)}(R) = 0$.*

Proof. See Corollary 3.7 in [32]. □

Now we apply these results.

Lemma 3.5. *Let B be a p -block of a finite group with metacyclic, minimal non-abelian defect group D for an odd prime p . Then every lower defect group of B is D -conjugate either to $\langle y \rangle$, $\langle y^p \rangle$, or to D .*

Proof. Let $R < D$ be a lower defect group of B . Then $m(|R|) > 0$ by Proposition 3.1. Corollary 5 in [36] shows that $p^{n-1} \mid |R|$. Since \mathcal{F} is controlled, the subgroup R is fully \mathcal{F} -centralized and fully \mathcal{F} -normalized. The fusion system of b_R (on $C_D(R)$) is given by $C_{\mathcal{F}}(R)$ (see Theorem IV.3.19 in [2]). Suppose for the moment that $C_{\mathcal{F}}(R)$ is trivial. Then b_R is nilpotent and $l(b_R) = 1$. Let $B_R := b_R^{N_G(R, b_R)}$. Then B_R has defect group $N_D(R)$ and $m_{B_R}^{(1)}(N_D(R)) = 1$. Hence, Lemmas 3.2 and 3.3 imply $m_B^{(1)}(R, b_R) = m_{B_R}^{(1)}(R) = 0$. This contradiction shows that $C_{\mathcal{F}}(R)$ is non-trivial. In particular R is centralized by a non-trivial p' -automorphism $\beta \in \text{Aut}_{\mathcal{F}}(D)$. By the Schur-Zassenhaus Theorem, β is $\text{Inn}(D)$ -conjugate to a power of α . Thus, R is D -conjugate to a subgroup of $\langle y \rangle$. The result follows. \square

Proposition 3.6. *Let B be a p -block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime p . Then $e(B) \leq l(B) \leq 2e(B) - 1$.*

Proof. Let

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, yxy^{-1} = x^{1+p^{m-1}} \rangle$$

be a defect group of B . We argue by induction on n . Let $n = 1$. By Proposition 2.2 we have $e(B) \leq l(B)$ and

$$k(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) (1 + p - p^{-1}).$$

Moreover, Theorem 3.2 in [38] gives

$$k(B) - l(B) = \frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)(p - 1)$$

Hence,

$$\begin{aligned} l(B) &= k(B) - (k(B) - l(B)) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) (1 + p - p^{-1}) - \frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} - e(B)(p - 1) \\ &= 2e(B) - \frac{1}{p} \left(e(B) - \frac{1}{e(B)} \right) - \frac{1}{e(B)}, \end{aligned}$$

and the claim follows in this case.

Now suppose $n \geq 2$. We determine the multiplicities of the lower defect groups by using Lemma 3.5. As usual $m(|D|) = 1$. Consider the subpair $(\langle y \rangle, b_y)$. By Lemmas 3.1 and 3.2 we have $m(p^n) = m_B^{(1)}(\langle y \rangle, b_y) = m_{B_y}^{(1)}(\langle y \rangle)$ where $B_y := b_y^{N_G(\langle y \rangle, b_y)}$. Since $N_D(\langle y \rangle) = C_D(y)$, it follows easily that $N_G(\langle y \rangle, b_y) = C_G(y)$ and $B_y = b_y$. By Theorem IV.3.19 in [2] the block b_y has defect group $C_D(y)$ and fusion system $C_{\mathcal{F}}(\langle y \rangle)$. In particular $e(b_y) = e(B)$. It is well-known that b_y dominates a block $\overline{b_y}$ of $C_G(y)/\langle y \rangle$ with cyclic defect group $C_D(y)/\langle y \rangle$ and $e(\overline{b_y}) = e(b_y) = e(B)$ (see [30, Theorem 5.8.11]). By Dade's Theorem on blocks with cyclic defect groups we obtain $l(b_y) = e(B)$. Moreover, the Cartan matrix of $\overline{b_y}$ has elementary divisors 1 and $|C_D(y)/\langle y \rangle|$ where 1 occurs with multiplicity $e(B) - 1$ (this follows for example from [13]). Therefore, the Cartan matrix of b_y has elementary divisors p^n and $|C_D(y)|$ where p^n occurs with multiplicity $e(B) - 1$. Since $\langle y \rangle \subseteq Z(C_G(y))$, Lemma 3.4 implies $m(p^n) = m_{b_y}^{(1)}(\langle y \rangle) = e(B) - 1$.

Now consider $(\langle u \rangle, b_u)$ where $u := y^p \in Z(D)$. Here b_u has defect group D . By the first part of the proof (the case $n = 1$) we obtain $l(b_u) = l(\overline{b_u}) \leq 2e(B) - 1$. As above we have $m(p^{n-1}) = m_B^{(1)}(\langle u \rangle, b_u) = m_{b_u}^{(1)}(\langle u \rangle)$. Since p^n occurs as elementary divisor of the Cartan matrix of b_u with multiplicity $e(B) - 1$ (see above), it follows that $m(p^{n-1}) = m_{b_u}^{(1)}(\langle u \rangle) \leq e(B) - 1$. Now $l(B)$ is the sum over the multiplicities of elementary divisors of the Cartan matrix of B which is at most $m(|D|) + m(\langle y \rangle) + m(\langle u \rangle) \leq 1 + e(B) - 1 + e(B) - 1 = 2e(B) - 1$. \square

The next proposition gives a reduction method.

Proposition 3.7. *Let $p > 2$, $m \geq 2$ and $e \mid p - 1$ be fixed. Suppose that $l(B) = e$ holds for every block B with defect group*

$$D = \langle x, y \mid x^{p^m} = y^p = 1, yxy^{-1} = x^{1+p^{m-1}} \rangle$$

and $e(B) = e$. Then every block B with $e(B) = e$ and defect group

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, yxy^{-1} = x^{1+p^{m-1}} \rangle$$

where $n \geq 1$ satisfies the following:

$$\begin{aligned} k_0(B) &= \left(\frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^n, & k_1(B) &= \frac{p^{m-1} - p^{m-2}}{e(B)} p^{n-1}, \\ k(B) &= \left(\frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)p \right) p^{n-1}, & l(B) &= e(B). \end{aligned}$$

Proof. We use induction on n . In case $n = 1$ the result follows from Theorem 3.2 in [38], Theorem 2.3 and Proposition 2.5.

Now let $n \geq 2$. Let \mathcal{R} be a set of representatives for the \mathcal{F} -conjugacy classes of elements of D . We are going to use Theorem 5.9.4 in [30]. For $1 \neq u \in \mathcal{R}$, b_u has metacyclic defect group $C_D(u)$ and fusion system $C_{\mathcal{F}}(\langle u \rangle)$. If $C_{\mathcal{F}}(\langle u \rangle)$ is non-trivial, $\alpha \in \text{Aut}_{\mathcal{F}}(D)$ centralizes a D -conjugate of u . Hence, we may assume that $u \in \langle y \rangle$ in this case. If $\langle u \rangle = \langle y \rangle$, then b_u dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with cyclic defect group $C_D(u)/\langle u \rangle$. Hence, $l(b_u) = l(\overline{b_u}) = e(B)$. Now suppose that $\langle u \rangle < \langle y \rangle$. Then by induction we obtain $l(b_u) = l(\overline{b_u}) = e(B)$. Finally assume that $C_{\mathcal{F}}(\langle u \rangle)$ is trivial. Then b_u is nilpotent and $l(b_u) = 1$. It remains to determine \mathcal{R} . The powers of y are pairwise non-conjugate in \mathcal{F} . As in the proof of Proposition 2.5, D has precisely $p^{n+m-3}(p^2 + p - 1)$ conjugacy classes. Let C be one of these classes which do not intersect $\langle y \rangle$. Assume $\alpha^i(C) = C$ for some $i \in \mathbb{Z}$ such that $\alpha^i \neq 1$. Then there are elements $u \in C$ and $w \in D$ such that $\alpha^i(u) = wuw^{-1}$. Hence $\gamma := w^{-1}\alpha^i \in N_G(D, b_D) \cap C_G(u)$. Since γ is not a p -element, we conclude that u is conjugate to a power of y which was excluded. This shows that no nontrivial power of α can fix C as a set. Thus, all these conjugacy classes split in

$$\frac{p^2 + p - p^{3-m} - 1}{e(B)} p^{n+m-3}$$

orbits of length $e(B)$ under the action of $\text{Out}_{\mathcal{F}}(D)$. Now Theorem 5.9.4 in [30] implies

$$k(B) - l(B) = \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B) \right) p^n - e(B).$$

By Proposition 3.6 it follows that

$$k(B) \leq \left(\frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)p \right) p^{n-1} + e(B) - 1. \quad (3.2)$$

By Proposition 2.2 the left hand side of (3.2) is divisible by p^{n-1} . Since $e(B) - 1 < p^{n-1}$, we obtain the exact value of $k(B)$. It follows that $l(B) = e(B)$. Finally, Theorem 2.3 and Proposition 2.5 give $k_i(B)$. \square

For $p = 3$, Proposition 3.6 implies $l(B) \leq 3$. Here we are able to determine all block invariants.

Theorem 3.8. *Let B be a non-nilpotent 3-block of a finite group with metacyclic, minimal non-abelian defect groups. Then*

$$\begin{aligned} k_0(B) &= \frac{3^{m-2} + 1}{2} 3^{n+1}, & k_1(B) &= 3^{m+n-3}, \\ k(B) &= \frac{11 \cdot 3^{m-2} + 9}{2} 3^{n-1}, & l(B) &= e(B) = 2 \end{aligned}$$

with the notation from (1.1).

Proof. By Proposition 3.7 it suffices to settle the case $n = 1$. Here the claim holds for $m \leq 3$ by Theorem 3.7 in [38]. We will extend the proof of this result in order to handle the remaining $m \geq 4$. Since B is non-nilpotent, we have $e(B) = 2$. By Theorem 2.3 we know $k_0(B) = (3^m + 9)/2$. By way of contradiction we may assume that $l(B) = 3$ and $k_1(B) = 3^{m-2} + 1$ (see Theorem 3.4 in [38]).

We consider the generalized decomposition numbers $d_{\chi\varphi_z}^z$ where $z := x^3 \in Z(D)$ and φ_z is the unique irreducible Brauer character of b_z . Let $d^z := (d_{\chi\varphi_z}^z : \chi \in \text{Irr}(B))$. By the orthogonality relations we have $(d^z, d^z) = 3^{m+1}$. As in [18, Section 4] we can write

$$d^z = \sum_{i=0}^{2 \cdot 3^{m-2} - 1} a_i \zeta_{3^{m-1}}^i$$

for integral vectors a_i and a primitive 3^{m-1} -th root of unity $\zeta_{3^{m-1}} \in \mathbb{C}$. Since z is \mathcal{F} -conjugate to z^{-1} , the vector d^z is real. Hence, the vectors a_i are linearly dependent. More precisely, it turns out that the vectors a_i are spanned by $\{a_j : j \in J\}$ for a subset $J \subseteq \{0, \dots, 2 \cdot 3^{m-2} - 1\}$ such that $0 \in J$ and $|J| = 3^{m-2}$.

Let q be the quadratic form corresponding to the Dynkin diagram of type $A_{3^{m-2}}$. We set $a(\chi) := (a_j(\chi) : j \in J)$ for $\chi \in \text{Irr}(B)$. Since the subsection (z, b_z) gives equality in Theorem 4.10 in [18], we have

$$k_0(B) + 9k_1(B) = \sum_{\chi \in \text{Irr}(B)} q(a(\chi))$$

for a suitable ordering of J . This implies $q(a(\chi)) = 3^{2h(\chi)}$ for $\chi \in \text{Irr}(B)$ where $h(\chi)$ is the height of χ . Moreover, if $a_0(\chi) \neq 0$, then $a_0(\chi) = \pm 3^{h(\chi)}$ by Lemma 3.6 in [38]. By Lemma 4.7 in [18] we have $(a_0, a_0) = 27$.

In the next step we determine the number β of 3-rational characters of height 1. Since $(a_0, a_0) = 27$, we have $\beta < 4$. On the other hand, the Galois group \mathcal{G} of $\mathbb{Q}(\zeta_{3^{m-1}}) \cap \mathbb{R}$ over \mathbb{Q} acts on d^z and the length of every non-trivial orbit is divisible by 3 (because \mathcal{G} is a 3-group). This implies $\beta = 1$, since $k_1(B) = 3^{m-2} + 1$.

In order to derive a contradiction, we repeat the argument with the subsection (x, b_x) . Again we get equality in Theorem 4.10, but this time for $k_0(B)$ instead of $k_0(B) + 9k_1(B)$. Hence, $d^x(\chi) = 0$ for characters $\chi \in \text{Irr}(B)$ of height 1. Again we can write $d^x = \sum_{i=0}^{2 \cdot 3^{m-1} - 1} \overline{a}_i \zeta_{3^m}^i$ where \overline{a}_i are integral vectors. Lemma 4.7 in [18] implies $(\overline{a}_0, \overline{a}_0) = 9$. Using Lemma 3.6 in [38] we also have $\overline{a}_0(\chi) \in \{0, \pm 1\}$. By Proposition 3.3 in [38] we have precisely three 3-rational characters $\chi_1, \chi_2, \chi_3 \in \text{Irr}(B)$ of height 0 (note that altogether we have four 3-rational characters). Then $a_0(\chi_i) = \pm \overline{a}_0(\chi_i) = \pm 1$ for $i = 1, 2, 3$. By [36, Section 1] we have $\lambda * \chi_i \in \text{Irr}_0(B)$ and $(\lambda * \chi_i)(u) = \chi_i(u) = d_{\chi_i \varphi_u}^u \varphi_u(1)$ for $\lambda \in \text{Irr}(D/\text{foc}(B)) \cong C_3$ and $u \in \{x, z\}$. Since this action on $\text{Irr}_0(B)$ is free, we have nine characters $\psi \in \text{Irr}(B)$ such that $a_0(\psi) = \pm \overline{a}_0(\psi) = \pm 1$. In particular $(a_0, \overline{a}_0) \equiv 1 \pmod{2}$. By the orthogonality relations we have $(d^z, d^{x^j}) = 0$ for all $j \in \mathbb{Z}$ such that $3 \nmid j$. Using Galois theory we get the final contradiction $0 = (d^z, \overline{a}_0) = (a_0, \overline{a}_0) \equiv 1 \pmod{2}$. \square

In the smallest case $D \cong C_9 \rtimes C_3$ of Theorem 3.8 even more information on B were given in Theorem 4.5 in [38].

Corollary 3.9. *Alperin's Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 3-block with metacyclic, minimal non-abelian defect groups.*

Proof. Let D be a defect group of B . Since B is controlled, Alperin's Weight Conjecture asserts that $l(B) = l(B_D)$ where B_D is a Brauer correspondent of B in $N_G(D)$. Since both numbers equal $e(B)$, the conjecture holds.

Now we prove the Ordinary Weight Conjecture in the form of [2, Conjecture IV.5.49]. Since $\text{Out}_{\mathcal{F}}(D)$ is cyclic, all 2-cocycles appearing in this version are trivial. Therefore the conjecture asserts that $k_i(B)$ only depends on \mathcal{F} and thus on $e(B)$. Since the conjecture is known to hold for the principal block of the solvable group $G = D \rtimes C_{e(B)}$, the claim follows. \square

We remark that Alperin's Weight Conjecture is also true for the abelian defect groups $D \cong C_{3^n} \times C_{3^m}$ where $n \neq m$ (see [41, 34]).

We observe another consequence for arbitrary defect groups.

Corollary 3.10. *Let B be a 3-block of a finite group with defect group D . Suppose that $D/\langle z \rangle$ is metacyclic, minimal non-abelian for some $z \in Z(D)$. Then Brauer's $k(B)$ -Conjecture holds for B , i. e. $k(B) \leq |D|$.*

Proof. Let (z, b_z) be a major subsection of B . Then b_z dominates a block $\overline{b_z}$ of $C_G(z)/\langle z \rangle$ with metacyclic, minimal non-abelian defect group $D/\langle z \rangle$. Hence, Theorem 3.8 implies $l(b_z) = l(\overline{b_z}) \leq 2$. Now the claim follows from Theorem 2.1 in [39]. \square

In the situation of Theorem 3.8 it is straight-forward to distribute $\text{Irr}(B)$ into families of 3-conjugate and 3-rational characters (cf. Proposition 3.3 in [38]). However, it is not so easy to see which of these families lie in $\text{Irr}_0(B)$.

Now we turn to $p = 5$.

Theorem 3.11. *Let B be a 5-block of a finite group with non-abelian defect group $C_{25} \rtimes C_{5^n}$ where $n \geq 1$. Then*

$$\begin{aligned} k_0(B) &= \left(\frac{4}{e(B)} + e(B) \right) 5^n, & k_1(B) &= \frac{4}{e(B)} 5^{n-1}, \\ k(B) &= \left(\frac{24}{e(B)} + 5e(B) \right) 5^{n-1}, & l(B) &= e(B). \end{aligned}$$

Proof. By Proposition 3.7 it suffices to settle the case $n = 1$. Moreover by Theorem 4.4 in [38] we may assume that $e(B) = 4$. Then by Theorem 2.3 above and Proposition 4.2 in [38] we have $k_0(B) = 25$, $1 \leq k_1(B) \leq 3$, $26 \leq k(B) \leq 28$ and $4 \leq l(B) \leq 6$. We consider the generalized decomposition numbers $d_{\chi\varphi_z}^z$ where $z := x^5 \in Z(D)$ and φ_z is the unique irreducible Brauer character of b_z . Since all non-trivial powers of z are \mathcal{F} -conjugate, the numbers $d_{\chi\varphi_z}^z$ are integral. Also, these numbers are non-zero, because (z, b_z) is a major subsection. Moreover, $d_{\chi\varphi_z}^z \equiv 0 \pmod{p}$ for characters $\chi \in \text{Irr}(B)$ of height 1 (see Theorem V.9.4 in [12]). Let $d^z := (d_{\chi\varphi_z}^z : \chi \in \text{Irr}(B))$. By the orthogonality relations we have $(d^z, d^z) = 125$. Suppose by way of contradiction that $k_1(B) > 1$. Then it is easy to see that $d_{\chi\varphi_z}^z = \pm 5$ for characters $\chi \in \text{Irr}(B)$ of height 1. By [36, Section 1], the numbers $d_{\chi\varphi_z}^z$ ($\chi \in \text{Irr}_0(B)$) split in five orbits of length 5 each. Let α (respectively β, γ) be the number of entries ± 1 (respectively $\pm 2, \pm 3$) in d^z . Then the orthogonality relations reads

$$\alpha + 4\beta + 9\gamma + 5k_1(B) = 25.$$

Since $\alpha + \beta + \gamma = 5$, we obtain

$$3\beta + 8\gamma = 20 - 5k_1(B) \in \{5, 10\}.$$

However, this equation cannot hold for any choice of α, β, γ . Therefore we have proved that $k_1(B) = 1$. Now Theorem 4.1 in [38] implies $l(B) = 4$. \square

Corollary 3.12. *Alperin's Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 5-block with non-abelian defect group $C_{25} \rtimes C_{5^n}$.*

Proof. See Corollary 3.9. \square

Unfortunately, the proof of Theorem 3.11 does not work for $p = 7$ and $e(B) = 6$ (even by invoking the other generalized decomposition numbers). However, we have the following partial result.

Proposition 3.13. *Let $p \in \{7, 11, 13, 17, 23, 29\}$ and let B be a p -block of a finite group with defect group $C_{p^2} \rtimes C_{p^n}$ where $n \geq 1$. If $e(B) = 2$, then*

$$\begin{aligned} k_0(B) &= \frac{p+3}{2} p^n, & k_1(B) &= \frac{p-1}{2} p^{n-1}, \\ k(B) &= \frac{p^2 + 4p - 1}{2} p^{n-1}, & l(B) &= 2. \end{aligned}$$

Proof. We follow the proof of Theorem 4.4 in [38] in order to handle the case $n = 1$. After that the result follows from Proposition 3.7.

In fact the first part of the proof of Theorem 4.4 in [38] applies to any prime $p \geq 7$. Hence, we know that the generalized decomposition numbers $d_{\chi\varphi_z}^z = a_0(\chi)$ for $z := x^p$ and $\chi \in \text{Irr}_0(B)$ are integral. Moreover,

$$\sum_{\chi \in \text{Irr}_0(B)} a_0(\chi)^2 = p^2.$$

The action of $D/\text{foc}(B)$ on $\text{Irr}_0(B)$ shows that the values $a(\chi)$ distribute in $(p+3)/2$ parts of p equal numbers each. Therefore, Eq. (4.1) in [38] becomes

$$\sum_{i=2}^{\infty} r_i(i^2 - 1) = \frac{p-3}{2}$$

for some $r_i \geq 0$. This gives a contradiction. □

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Benjamin Sambale
Institut für Mathematik
Friedrich-Schiller-Universität
07743 Jena
Germany
benjamin.sambale@uni-jena.de